

SATELLITE MOTION AROUND AN OBLATE PLANET: A PERTURBATION SOLUTION FOR ALL ORBITAL PARAMETERS

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Abstract

The search for a universal solution of the equations of motion for a satellite orbiting an oblate planet is a subject that has merited great interest because of its theoretical and practical implications. Here, a complete first-order perturbation solution, including the effects of the J_2 terms in the planet's potential, is given in terms of standard orbital parameters. The simple formulas provide a fast method for predicting satellite orbits that is more accurate than the two-body formulas. These predictions are shown to agree well with those of a completely numerical code and with actual satellite data. Also, in an appendix, it is rigorously proven that a satellite having negative mechanical energy remains for all time within a spherical annulus with radii approximately equal to the perigee and apogee of its initial osculating ellipse.

1 Introduction

A characteristic feature of practical orbit prediction is that the engineer may deal with numerous satellites in a great variety of orbits. Under these circumstances analytical relations which can quickly approximate an orbit may be far superior to large numerical programs. While many analytical models have been developed for the artificial satellite age, most are not used in practical orbit prediction because they violate one or more of the following principles:

- The method should provide a solution that is significantly more accurate than the two-body solution.
- The real physical effects of the orbit should be easily distinguishable in the solution.
- The solution should be universal; it should be valid for all orbital parameters.

The problem of predicting the motion of a satellite perturbed only by the oblateness of the planet has received considerable attention following the first launchings of artificial satellites about the Earth. Some of the studies of this problem by means of general perturbation theories are listed at the end of this paper. Techniques have involved expansions in powers of $\sqrt{J_2}$, averaging processes, the use of spheroidal coordinates, and the edifice of Hamiltonian mechanics. It is not the intention of this present paper to compare the various methodologies used. Suffice it to say that many researchers believe a solution which embodies all of the above principles was not achieved (e.g., see Taff).

The basic procedure used in this paper to solve the differential equations of motion is the perturbation technique known as the Method of Strained Coordinates. This technique was first applied to the title problem by Brenner, Latta, and Weisfield. Using a *mean* orbital plane to specify an arbitrary orbit, they were only able to obtain a partial solution (e.g., the eccentricity was assumed small and initial conditions were not considered).

Here we use coordinates in the *true* orbital plane to cast the differential equations into a simplified form, as was originally done by Struble.

2 Orbital Kinematics

Figure 1 shows the usual reference system of spherical coordinates (r, α, β) . The radial distance r is measured from the center of the planet O to the satellite S . The line $O\gamma$ is in a direction fixed with respect to an inertial coordinate system. The right ascension α is the angle measured in the planet's equatorial plane eastward from the line $O\gamma$. The declination or latitude β is the angle measured northward from the equator. The position vector \mathbf{r} of the satellite in the spherical coordinate system is

$$\mathbf{r} = r(\cos \alpha \cos \beta)\mathbf{b}_1 + r(\sin \alpha \cos \beta)\mathbf{b}_2 + r(\sin \beta)\mathbf{b}_3 \quad (1)$$

where $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ are orthonormal base vectors fixed in the directions shown.

We can also locate the satellite by its polar coordinates (r, θ) within a (possibly rotating) orbital plane that instantaneously contains its position and velocity vectors. Here θ is the argument of latitude, i.e., the angle measured in the orbital plane from the ascending node to the satellite. The orbital plane is inclined at an angle i to the equatorial plane and intersects the equatorial plane in the line of nodes, making an angle Ω with the $O\gamma$ line.

We introduce another orthonormal set of base vectors $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ which move with the satellite so that \mathbf{B}_1 is in the direction of the position vector \mathbf{r} , \mathbf{B}_2 is also in the orbital plane, and $\mathbf{B}_3 = \mathbf{B}_1 \times \mathbf{B}_2$. The basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ may be transformed into the basis $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ by a succession of three rotations. First the basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ is rotated about the \mathbf{b}_3 direction by the angle Ω , next the basis is rotated about the new 1-direction by the angle i , and finally the basis is again rotated about the new 3-direction by the angle θ . The two sets of base vectors are related by the product of the rotation matrices representing each successive

rotation (as explained in the book by Danielson):

$$\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos \Omega & \sin \Omega & 0 \\ -\sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} \quad (2)$$

or

$$\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \end{bmatrix} = \begin{bmatrix} \cos \theta \cos \Omega - \sin \theta \cos i \sin \Omega & \cos \theta \sin \Omega + \sin \theta \cos i \cos \Omega & \sin \theta \sin i \\ -\sin \theta \cos \Omega - \cos \theta \cos i \sin \Omega & -\sin \theta \sin \Omega + \cos \theta \cos i \cos \Omega & \cos \theta \sin i \\ \sin i \sin \Omega & -\sin i \cos \Omega & \cos i \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$

The position vector \mathbf{r} has only one component in the rotating basis:

$$\mathbf{r} = r\mathbf{B}_1 \quad (3)$$

Using the first of equations (2), we obtain the components of \mathbf{r} in the fixed basis:

$$\begin{aligned} \mathbf{r} = & r(\cos \theta \cos \Omega - \sin \theta \cos i \sin \Omega)\mathbf{b}_1 \\ & + r(\cos \theta \sin \Omega + \sin \theta \cos i \cos \Omega)\mathbf{b}_2 + r(\sin \theta \sin i)\mathbf{b}_3 \end{aligned} \quad (4)$$

Equating the components of equations (1) and (4), we can obtain the following relations among the angles (α, β) of the spherical coordinate system and the astronomical angles (i, Ω, θ) :

$$\sin \beta = \sin \theta \sin i \quad (5)$$

$$\cos \beta = \cos \theta \sec(\alpha - \Omega)$$

The velocity $d\mathbf{r}/dt$ of the satellite is obtained by differentiating (3) with respect to the time t :

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{B}_1 + r\frac{d\mathbf{B}_1}{dt} \quad (6)$$

Since the orbital plane must contain the velocity vector, we have to enforce

$$\frac{d\mathbf{B}_1}{dt} \cdot \mathbf{B}_3 = 0 \quad (7)$$

Substitution of equations (2) into equation (7) leads to a relationship which uncouples the equations for $\Omega(\theta)$ and $i(\theta)$:

$$\frac{d\Omega}{d\theta} = \frac{\tan \theta}{\sin i} \frac{di}{d\theta} \quad (8)$$

The velocity (6) can then be written

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{B}_1 + r\frac{d\theta}{dt}\left(1 + \tan\theta \cot i \frac{di}{d\theta}\right)\mathbf{B}_2 \quad (9)$$

In the following part of this paper, we will obtain expressions for $r(\theta)$, $i(\theta)$, $\Omega(\theta)$, and $dt/d\theta(\theta)$. The position and velocity vectors of the satellite then may be calculated from the formulas in this section. The classical orbital elements p, e , and ω are the semilatus rectum, eccentricity, and argument of perigee of the instantaneous (osculating) conic section determined by the position and velocity vectors. If needed, $p(\theta)$, $e(\theta)$, and $\omega(\theta)$ can be obtained from our solution $r(\theta)$ and $dt/d\theta(\theta)$:

$$p = \frac{r^4}{GM \left(\frac{dt}{d\theta}\right)^2}$$

$$e \cos(\theta - \omega) = \frac{p}{r} - 1$$

$$e \sin(\theta - \omega) = \frac{p}{r^2} \left(\frac{dr}{d\theta}\right)$$

3 Equations of Motion

The expressions for the kinetic and potential energies per unit mass of a satellite orbiting around an oblate planet are respectively:

$$T = \frac{1}{2} \left[\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\beta}{dt}\right)^2 + r^2 \cos^2 \beta \left(\frac{d\alpha}{dt}\right)^2 \right] \quad (10)$$

$$V = -\frac{GM}{r} \left[1 + \frac{J_2 R^2}{2r^2} (1 - 3 \sin^2 \beta) \right] \quad (11)$$

where G is the gravitational constant, M is the mass of the planet, R is the equatorial radius of the planet, and J_2 is the constant coefficient of the spherical harmonic of degree 2 and order 0 in the planet's gravitational field. Substitution of these equations into Lagrange's equations

$$\frac{d}{dt} \frac{\partial(T - V)}{\partial \left(\frac{dq}{dt}\right)} - \frac{\partial}{\partial q}(T - V) = 0 \quad q = r, \alpha, \text{ or } \beta$$

results in the following equations of motion:

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\beta}{dt} \right)^2 - r \cos^2 \beta \left(\frac{d\alpha}{dt} \right)^2 = -\frac{\partial V}{\partial r} \quad (12)$$

$$\begin{aligned} \frac{d}{dt} \left(r^2 \cos^2 \beta \frac{d\alpha}{dt} \right) &= 0 \\ \frac{d}{dt} \left(r^2 \frac{d\beta}{dt} \right) + r^2 \sin \beta \cos \beta \left(\frac{d\alpha}{dt} \right)^2 &= -\frac{\partial V}{\partial \beta} \end{aligned} \quad (13)$$

Initial conditions are established by requiring that at the initial time t_0 the orbital parameters of the usual two-body orbit, the conic section determined by the initial position and velocity vectors, are known. The actual orbit is then tangent to this initial instantaneous conic section at t_0 (see Figure 1). Equating the initial position and velocity vectors given by equations (3) and (9) to the two-body expressions, we obtain

$$r(t_0) = \frac{p_0}{1 + e_0 \cos(\theta_0 - \omega_0)}, \quad (14)$$

$$\frac{dr}{dt}(t_0) = \frac{e_0 h_0 \sin(\theta_0 - \omega_0)}{p_0} \quad (15)$$

$$\frac{d\theta}{dt}(t_0) = \frac{h_0}{r_0^2 \left[1 + \tan \theta_0 \cot i_0 \frac{di}{d\theta}(\theta_0) \right]} \quad (16)$$

$$i(\theta_0) = i_0 \quad (17)$$

$$\Omega(\theta_0) = \Omega_0 \quad (18)$$

Here $h_0 = \sqrt{GMp_0}$ is the initial value of the satellite's specific angular momentum about the center of the planet, and the subscript 0 on a symbol denotes that the parameter is evaluated at the initial time t_0 .

We immediately have two integrals of the equations of motion:

$$T + V = \text{constant} \quad (19)$$

$$r^2 \cos^2 \beta \frac{d\alpha}{dt} = \text{constant} \quad (20)$$

Equation (19) simply states that the mechanical energy of the satellite remains constant. Now, from equations (1) and (16)

$$r^2 \cos^2 \beta \frac{d\alpha}{dt} = \mathbf{r} \times \frac{d\mathbf{r}}{dt} \cdot \mathbf{b}_3 = h_0 \cos i_0 \quad (21)$$

Equation (21) simply states that the component along the polar axis of the specific angular momentum of the satellite remains constant. Inserting equations (3) and (9) into equation (21), we obtain

$$\frac{dt}{d\theta} = \frac{r^2 \cos i}{h_0 \cos i_0} \left(1 + \tan \theta \cot i \frac{di}{d\theta} \right) \quad (22)$$

This allows the independent variable to be changed from t to θ .

Letting $u = p_0/r$, and using equations (5), (21), and (22), we can rewrite the remaining equations of motion (12)–(13):

$$\frac{di}{d\theta} = \frac{-2Ju \sin \theta \cos \theta \sin i \cos^2 i}{\frac{c^2}{\cos i} + 2Ju \sin^2 \theta \cos^3 i} \quad (23)$$

$$\begin{aligned} \frac{d^2 u}{d\theta^2} + u &= \frac{\cos^2 i}{c^2} + \frac{J \cos^2 i}{c^2} \left[u^2 (1 - 3 \sin^2 \theta \sin^2 i) + 2u \frac{du}{d\theta} \sin \theta \cos \theta (1 - 3 \cos^2 i) \right. \\ &\quad \left. - 4u \frac{d^2 u}{d\theta^2} \sin^2 \theta \cos^2 i - 2 \left(\frac{du}{d\theta} \right)^2 \sin^2 \theta \cos^2 i \right] - \frac{4J^2 u \sin^3 \theta \cos^6 i}{c^4} \\ &\quad \left[u \frac{du}{d\theta} \cos \theta (2 + \sin^2 i) + \frac{d}{d\theta} \left(u \frac{du}{d\theta} \right) \sin \theta \cos^2 i \right] \end{aligned} \quad (24)$$

The terms in (24) with $d^2 u/d\theta^2$ can be combined, yielding the equivalent equation

$$\begin{aligned} \frac{d^2 u}{d\theta^2} + u &= \left\{ \frac{\cos^2 i}{c^2} + \frac{J \cos^2 i}{c^2} [u^2 (1 + \sin^2 \theta (7 \cos^2 i - 3)) \right. \\ &\quad \left. + 2u \frac{du}{d\theta} \sin \theta \cos \theta (1 - 3 \cos^2 i) - 2 \left(\frac{du}{d\theta} \right)^2 \sin^2 \theta \cos^2 i] \right. \\ &\quad \left. + \frac{4J^2 u \sin^3 \theta \cos^6 i}{c^4} \left[u^2 \sin \theta \cos^2 i - u \frac{du}{d\theta} \cos \theta (2 + \sin^2 i) - \left(\frac{du}{d\theta} \right)^2 \sin \theta \cos^2 i \right] \right\} \\ &\quad \div \left(1 + \frac{4Ju \sin^2 \theta \cos^4 i}{c^2} + \frac{4J^2 u^2 \sin^4 \theta \cos^8 i}{c^4} \right) \end{aligned} \quad (25)$$

Here we have introduced the shorthand notation $c = \cos i_0$, $s = \sin i_0$, $J = 3J_2R^2/2p_0^2$.

4 Perturbation Procedure

The differential equations (23)–(24) are coupled by the nonlinear terms and apparently cannot be solved analytically. If we expand the right sides of (23) and (25) in a Taylor series expansion in powers of J and retain only terms up to order J^2 , the equations simplify to

$$\frac{di}{d\theta} = \frac{-2Ju \sin \theta \cos \theta \sin i \cos^3 i}{c^2} + \frac{4J^2u^2 \sin i \cos^7 i}{c^4} \sin^3 \theta \cos \theta + O(J^3) \quad (26)$$

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u = & \frac{\cos^2 i}{c^2} + \frac{J \cos^2 i}{c^2} \left\{ \frac{-4u \sin^2 \theta \cos^4 i}{c^2} + u^2[1 + \sin^2 \theta(7 \cos^2 i - 3)] \right. \\ & \left. + 2u \frac{du}{d\theta} \sin \theta \cos \theta(1 - 3 \cos^2 i) - 2 \left(\frac{du}{d\theta} \right)^2 \sin^2 \theta \cos^2 i \right\} \\ & + \frac{4J^2u \sin^2 \theta \cos^6 i}{c^4} \left\{ u^2[-1 + 3 \sin^2 \theta(1 - 2 \cos^2 i)] + \frac{3u \sin^2 \theta \cos^4 i}{c^2} \right. \\ & \left. + u \frac{du}{d\theta} \sin \theta \cos \theta[7 \cos^2 i - 5] + \left(\frac{du}{d\theta} \right)^2 \sin^2 \theta \cos^2 i \right\} + O(J^3) \end{aligned} \quad (27)$$

Here the term in the O symbols indicates that, for all sufficiently small J , the error is less than a constant times J^3 . The equations (26)–(27) are identical to those used as the starting point in the analysis of Eckstein, et al.

It is reasonable to expect that the solution for u will be arbitrarily close to the two body solution, $1 + e_0 \cos(\theta - \omega_0)$, when J is close to zero. This assumption is consistent with letting

$$u = 1 + e_0 \cos y + Ju_1 + J^2u_2 + \dots \quad (28)$$

$$y = \theta - \omega_0 + Jy_1 + J^2y_2 + \dots \quad (29)$$

$$i = i_0 + Ji_1 + J^2i_2 + \dots \quad (30)$$

An algorithm for the perturbation procedure is:

Let $n = 1$

Substitute expressions (28)–(30) into the equations of motion (26)–(27)

Equate the coefficients of J^n

Choose the arbitrary constants so secular terms will not arise.

Solve for the n^{th} order solution

Satisfy the initial conditions (14)–(18)

Iterate on n

The calculations were carried out with the symbolic manipulation program MACSYMA. In this paper we only briefly outline these calculations; for more details see the theses of Sagovac and Snider.

Beginning by substituting equations (28) and (30) into (26), and equating the terms multiplied by J , we obtain

$$\frac{di_1}{d\theta} = -sc \sin 2\theta - \frac{sce_0}{2} \sin(y + 2\theta) + \frac{sce_0}{2} \sin(y - 2\theta) \quad (31)$$

A solution to this equation is

$$i_1 = \frac{sc}{2} \cos 2\theta + \frac{sce_0}{6} \cos(y + 2\theta) + \frac{sce_0}{2} \cos(y - 2\theta) + K_1 \cos(2y - 2\theta) + K_2 \quad (32)$$

The last two terms may be added because they are to lowest order homogenous solutions to equation (30). The term multiplied by the constant K_1 was added to eliminate secular terms in i_2 ; note that differentiating this term with respect to θ produces terms multiplied by J , from equation (29). The constant K_2 was added to satisfy the initial condition (17), which implies that $i_1(\theta_0) = 0$ so

$$K_2 = -\frac{sc}{2} \cos 2\theta_0 - \frac{sce_0}{6} \cos(3\theta_0 - \omega_0) - \frac{sce_0}{2} \cos(\theta_0 + \omega_0) - K_1 \cos 2\omega_0$$

Substituting equations (28)–(30) and (32) into (27), and equating terms multiplied by J yields

$$\begin{aligned} \frac{d^2 u_1}{d\theta^2} + u_1 = & 1 - \frac{3s^2}{2} + e_0^2 \left(-\frac{5s^2}{4} + 1 \right) + \frac{1}{4} [(2 + 5e_0^2)s^2 - 2e_0^2] \cos 2\theta \\ & + \frac{e_0^2}{4} (-9s^2 + 8) \cos 2y + \frac{e_0}{3} (11s^2 - 6) \cos(y + 2\theta) + \frac{15e_0^2}{24} (3s^2 - 2) \cos(2y + 2\theta) \end{aligned} \quad (33)$$

$$+ \left[\frac{e_0^2}{8}(3s^2 - 2) - \frac{2sK_1}{c} \right] \cos(2y - 2\theta) - \frac{2sK_2}{c} + e_0 \left(2\frac{dy_1}{d\theta} + 4 - 5s^2 \right) \cos y + e_0 \frac{d^2 y_1}{d\theta^2} \sin y$$

In the above equation, the $\cos y$ and $\sin y$ terms would produce secular terms $\theta \sin y$ and $\theta \cos y$ in u_1 . The choice $dy_1/d\theta = 5s^2/2 - 2$ will eliminate these possibilities. Integrating yields

$$y_1 = \left(\frac{5s^2}{2} - 2 \right) (\theta - \theta_0) + K_3 [\sin(2y - 2\theta) + \sin 2\omega_0] \quad (34)$$

The term multiplied by K_3 was added to eliminate secular terms in u_2 . The constant terms in (34) were added to satisfy the initial condition $y(\theta_0) = \theta_0 - \omega_0$.

A solution to lowest order of equation (33) is then

$$\begin{aligned} u_1 = & 1 - \frac{3s^2}{2} + e_0^2 \left(\frac{-5s^2}{4} + 1 \right) + \frac{1}{12} [-s^2(2 + 5e_0^2) + 2e_0^2] \cos 2\theta \\ & + \frac{e_0^2}{12} (9s^2 - 8) \cos 2y + \frac{e_0}{24} (-11s^2 + 6) \cos(y + 2\theta) + \frac{e_0^2}{24} (-3s^2 + 2) \cos(2y + 2\theta) \\ & + \left[\frac{e_0^2}{8} (3s^2 - 2) - \frac{2sK_1}{c} \right] \cos(2y - 2\theta) - \frac{2sK_2}{c} + K_4 \cos(y - 2\theta) \\ & + K_5 \cos(y - \theta_0 + \omega_0) + K_6 \sin(y - \theta_0 + \omega_0) \end{aligned} \quad (35)$$

The term multiplied by K_4 was added to eliminate secular terms in u_2 . The terms multiplied by K_5 and K_6 were added to satisfy the initial conditions (14)–(16).

With all terms in place to deal with secular terms, the calculations are continued by substituting equations (28)–(30), (32), (34), and (35) into (26) and equating terms multiplied by J^2 :

$$\frac{di_2}{d\theta} = \left[K_1 + \frac{sce_0^2(15s^2 - 14)}{24(5s^2 - 4)} \right] \sin(2y - 2\theta) + \dots \quad (36)$$

We have for brevity only indicated on the right side of equation (36) the term that would produce secular terms in i_2 . Removal of this term by making its coefficient zero determines K_1 . Equation (36) is then integrated to determine i_2 .

Continuing the procedure by equating the terms multiplied by J^2 in the expansion of equation (27) determines y_2 , K_3 , and K_4 . Final values of all the constants are listed in Appendix I.

Knowing the solution for $i(\theta)$, we can determine $\Omega(\theta)$ by integrating equation (8) and applying the initial condition (18). The angle θ , which increases continuously from an initial value θ_0 , may be related to the time t by numerically integrating (22).

5 Solution

Here we assemble the complete solution:

$$\begin{aligned}
r = & p_0 / \left\{ 1 + e_0 \cos y + J \left[1 - \frac{3s^2}{2} + e_0^2 \left(1 - \frac{5s^2}{4} \right) + \frac{1}{12} (-(2 + 5e_0^2)s^2 + 2e_0^2) \cos 2\theta \right. \right. \\
& + \frac{e_0^2}{12} (9s^2 - 8) \cos 2y + \frac{e_0}{24} (-11s^2 + 6) \cos(y + 2\theta) + \frac{e_0^2}{24} (-3s^2 + 2) \cos(2y + 2\theta) \\
& + \frac{e_0^2}{8} (3s^2 - 2) \cos(2y - 2\theta) \\
& + \frac{e_0 [15(2 + e_0^2)s^4 - 14(4 + e_0^2)s^2 + 24] \sin \left[\frac{J\theta}{2} (5s^2 - 4) \right] \sin[\theta + \omega_0]}{12(5s^2 - 4)} \\
& + \frac{e_0^2 s^2 (15s^2 - 14) \sin \left[\frac{J\theta}{2} (5s^2 - 4) \right] \sin \left[2\omega_0 - \frac{J\theta}{2} (5s^2 - 4) \right]}{6(5s^2 - 4)} - \frac{e_0^2 s^2}{16} \cos(y - \theta_0 + 3\omega_0) \\
& + \frac{e_0^2}{24} (3s^2 - 2) \cos(y - 3\theta_0 + 3\omega_0) - \frac{e_0^2 s^2}{16} \cos(y - 5\theta_0 + 3\omega_0) \\
& + \frac{e_0}{4} (3s^2 - 2) \cos(y - 2\theta_0 + 2\omega_0) - \frac{3e_0 s^2}{8} \cos(y - 4\theta_0 + 2\omega_0) \\
& - \frac{e_0}{4} (s^2 + 1) \cos(y + 2\omega_0) + \frac{1}{8} [(-2 + 5e_0^2)s^2 - 2e_0^2] \cos(y + \theta_0 + \omega_0) \\
& + \frac{1}{4} [(6 + 5e_0^2)s^2 - 4(1 + e_0^2)] \cos(y - \theta_0 + \omega_0) \\
& + \frac{1}{24} [-(14 + 5e_0^2)s^2 + 2e_0^2] \cos(y - 3\theta_0 + \omega_0) \\
& + \frac{e_0^2}{48} (9s^2 - 4) \cos(y + 3\theta_0 - \omega_0) + \frac{e_0^2}{8} (-7s^2 + 6) \cos(y + \theta_0 - \omega_0) \\
& + \frac{e_0^2}{16} (-5s^2 + 4) \cos(y - \theta_0 - \omega_0) \\
& + \frac{e_0}{4} (2s^2 - 1) \cos(y + 2\theta_0) + \frac{e_0}{4} (-3s^2 + 1) \cos(y - 2\theta_0) + \frac{e_0}{4} (-3s^2 + 2) \cos y \\
& \left. + e_0 s^2 \cos(\theta_0 + \omega_0) + \frac{e_0 s^2}{3} \cos(3\theta_0 - \omega_0) + s^2 \cos 2\theta_0 \right\} + p_0 O(J^2, J^3 \theta)
\end{aligned} \tag{37}$$

$$\begin{aligned}
y &= \theta - \omega_0 + J \left(\frac{5s^2}{2} - 2 \right) (\theta - \theta_0) \\
&+ \frac{J e_0^2}{24(5s^2 - 4)} \left\{ \frac{(-75s^6 + 260s^4 - 296s^2 + 112) \sin \left[\frac{J\theta}{2} (5s^2 - 4) \right] \cos \left[2\omega_0 - \frac{J\theta}{2} (5s^2 - 4) \right]}{(5s^2 - 4)} \right. \\
&+ J\theta s^2 (-15s^2 + 14)(15s^2 - 13) \cos 2\omega_0 \left. \right\} + J^2 \theta \left\{ \frac{e_0 s^2}{2} (15s^2 - 13) \cos(\theta_0 + \omega_0) \right. \\
&+ \frac{e_0 s^2}{6} (15s^2 - 13) \cos(3\theta_0 - \omega_0) + \frac{s^2}{2} (15s^2 - 13) \cos 2\theta_0 \\
&+ \frac{1}{96} [5(9e_0^2 + 34)s^4 + 4(9e_0^2 - 34)s^2 - 56e_0^2] \left. \right\} + O(J^2, J^3\theta)
\end{aligned} \tag{38}$$

$$\begin{aligned}
i &= i_0 + s c J \left\{ \frac{1}{2} \cos 2\theta + \frac{e_0}{6} \cos(y + 2\theta) \right. \\
&+ \frac{e_0}{2} \cos(y - 2\theta) + \frac{e_0^2 (-15s^2 + 14) \sin \left[\frac{J\theta}{2} (5s^2 - 4) \right] \sin \left[2\omega_0 - \frac{J\theta}{2} (5s^2 - 4) \right]}{12(5s^2 - 4)} \\
&- \frac{1}{2} \cos 2\theta_0 - \frac{e_0}{6} \cos(3\theta_0 - \omega_0) - \frac{e_0}{2} \cos(\theta_0 + \omega_0) \left. \right\} + O(J^2, J^3\theta)
\end{aligned} \tag{39}$$

$$\begin{aligned}
\Omega &= \Omega_0 + c J \left[\theta_0 - \theta + \frac{1}{2} \sin 2\theta - e_0 \sin y + \frac{e_0}{6} \sin(y + 2\theta) - \frac{e_0}{2} \sin(y - 2\theta) - \frac{1}{2} \sin 2\theta_0 \right. \\
&+ e_0 \sin(\theta_0 - \omega_0) - \frac{e_0}{6} \sin(3\theta_0 - \omega_0) - \frac{e_0}{2} \sin(\theta_0 + \omega_0) \left. \right] \\
&+ \frac{c J e_0^2}{12(5s^2 - 4)} \left\{ \frac{2(15s^4 - 45s^2 + 28) \sin \left[\frac{J\theta}{2} (5s^2 - 4) \right] \cos \left[2\omega_0 - \frac{J\theta}{2} (5s^2 - 4) \right]}{(5s^2 - 4)} \right. \\
&+ J\theta s^2 (15s^2 - 14) \cos 2\omega_0 \left. \right\} + c J^2 \theta \left[-e_0 s^2 \cos(\theta_0 + \omega_0) - \frac{e_0 s^2}{3} \cos(3\theta_0 - \omega_0) \right. \\
&- s^2 \cos 2\theta_0 + \frac{e_0^2}{24} (7s^2 - 4) + \frac{1}{12} (-s^2 + 6) \left. \right] k + O(J^2, J^3\theta)
\end{aligned} \tag{40}$$

$$t = t_0 + \frac{1}{h_0} \int_{\theta_0}^{\theta} r^2 \left\{ 1 + J \left[\frac{(-3s^2 + 2)}{2} \cos 2\theta + e_0 (s^2 - 1) \right. \right.$$

$$\begin{aligned}
& \cos y + \frac{e_0(-4s^2 + 3)}{6} \cos(y + 2\theta) + \frac{e_0(-2s^2 + 1)}{2} \cos(y - 2\theta) \\
& + \frac{e_0^2 s^2 (15s^2 - 14) \sin \left[\frac{J\theta}{2} (5s^2 - 4) \right] \sin \left[2\omega_0 - \frac{J\theta}{2} (5s^2 - 4) \right]}{12(5s^2 - 4)} \\
& + s^2 - 1 + \frac{s^2}{2} \cos 2\theta_0 + \frac{e_0 s^2}{6} \cos(3\theta_0 - \omega_0) + \frac{e_0 s^2}{2} \cos(\theta_0 + \omega_0) \Big] \Big\} d\theta + \frac{p_0^2}{h_0} O(J^2, J^2\theta)
\end{aligned} \tag{41}$$

In obtaining the equations (37)–(41), use has been made of trigonometric formulas to simplify terms containing the factor $5s^2 - 4$ in the denominator. In the form given, these terms can clearly be seen to approach a finite limit at the “critical inclination” $i_0 = \sin^{-1} \sqrt{4/5} = 63^\circ 26'$ or $116^\circ 34'$. Hence the solution is actually valid for all values of i_0 . If $|i_0 - \sin^{-1} \sqrt{4/5}| < J$, the formulas (37)–(41) can still be used by letting $5s^2 - 4 = J$, or the limiting forms for $i_0 \rightarrow \sin^{-1} \sqrt{4/5}$ can be used.

To check the solution, we can see if the specific mechanical energy (18) of the satellite remains constant. Substitution of the solution (36)–(37) into equation (10) plus (11) yields

$$T + V = -\frac{GM(1 - e_0^2)}{2p_0} - \frac{GMJ_2R^2(1 - 3\sin^2 \beta_0)}{2r_0^3} + \frac{GM}{p_0} O(J^2)$$

The right side is easily recognized as the value of the specific mechanical energy at the initial time t_0 .

As a further check on the solution, we can see if it reduces to our previous results for equatorial and polar orbits, obtained by completely separate derivations (Danielson and Snider, 1989). Setting $i_0 = 0$ and using the independent variable α measured from the line $O\gamma$, we find that equations (37)–(41) reduce to equations (18)–(22) of our previous paper. Setting $i_0 = \pi/2$ and using the expansion $\cos(y + Jk) \approx \cos y - Jk \sin y$, we find that equations (37)–(41) reduce to equations (38)–(41) of our previous paper.

Comparing the terms in the O -symbols, we see that the relative error in equation (41) may be greater than that of equations (37)–(40). Since the underlined terms in equations (37)–(40) are of this same order of magnitude, we can drop the underlined terms except when (37)–(38) are used to calculate r in equation (41). The relative error of our solution

will then still be of order $(\theta - \theta_0)J^2$.

If we retain only the two-body solution, the relative error terms will be of the order $(\theta - \theta_0)J$. Here the error in our solution, as compared to the exact solution of the equations of motion, should be of the order J times the error in the two-body solution (for an Earth satellite $J < .0015$).

6 Comparison of Perturbation, Two-Body, Numerical, and Measured Solutions

In this section we compare the preceding perturbation solution, the two-body solution, a completely numerical solution of the differential equations, and actual measured satellite data; for more comparisons see the thesis of Krambeck. The difference between the position vector \mathbf{r} determined by the numerical integration code or measured data and the position vector \mathbf{r}_{ref} calculated from our perturbation solution or the two-body solution is the error $\Delta\mathbf{r}$:

$$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_{\text{ref}}$$

If the errors $(\Delta r, \Delta\theta, \Delta i, \Delta\Omega)$ in the orbital parameters (r, θ, i, Ω) are small, we can estimate $\Delta\mathbf{r}$ from equation (4) and the linear approximation

$$\Delta\mathbf{r} \approx \frac{\partial\mathbf{r}}{\partial r}\Delta r + \frac{\partial\mathbf{r}}{\partial\theta}\Delta\theta + \frac{\partial\mathbf{r}}{\partial i}\Delta i + \frac{\partial\mathbf{r}}{\partial\Omega}\Delta\Omega \quad (42)$$

It is customary to decompose $\Delta\mathbf{r}$ into components $(\delta_1, \delta_2, \delta_3)$ along the moving triad $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$:

$$\Delta\mathbf{r} = \delta_1\mathbf{B}_1 + \delta_2\mathbf{B}_2 + \delta_3\mathbf{B}_3$$

The component δ_1 is called the radial error, δ_2 is the down track error, and δ_3 is the cross track error. Applying (42) to equation (4), and expressing the base vectors $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ in terms of $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$, we obtain the following approximations:

$$\delta_1 \approx \Delta r, \quad \delta_2 \approx r(\Delta\theta + \cos i\Delta\Omega), \quad \delta_3 \approx r(\sin\theta\Delta i - \cos\theta\sin i\Delta\Omega) \quad (43)$$

We obtained the numerical integration code UTOPIA from the Colorado Center for Astrodynamics Research located on the campus of the University of Colorado. The code was specialized to the differential equations used in this paper. We compared the solutions for an earth satellite with the following initial conditions:

$$r_0 = 7,386.18 \text{ km}$$

$$e_0 = .003991$$

$$\theta_0 = 104.05^\circ$$

$$\omega_0 = 224.38^\circ$$

$$i_0 = 90.03^\circ$$

$$\Omega_0 = 322.63^\circ$$

$$t_0 = 0$$

These initial conditions represent an essentially polar orbit at an altitude of approximately 1000 kilometers and period about $1\frac{3}{4}$ hours. For this satellite the perturbation and numerical orbits match extremely well while the two-body orbit is grossly erroneous. The magnitude of the error in \mathbf{r} is shown in Figure 2. Note that the relative error in our perturbation solution is $2.8J^2(\theta - \theta_0)$, and that this error is $1.1J$ times the error in the two-body solution.

We obtained measured satellite data from the First Satellite Control Squadron located at Falcon Air Force Base, Colorado. A near earth satellite processed the following initial conditions:

$$r_0 = 7,776.58 \text{ km}$$

$$e_0 = .0003071$$

$$\theta_0 = 149.14^\circ$$

$$\omega_0 = 9.57^\circ$$

$$i_0 = 98.81^\circ$$

$$\Omega_0 = 37.10^\circ$$

$$t_0 = 0000Z \text{ 26 July 1990}$$

Again, the perturbation orbit is far superior to the two-body orbit. The radial, down track, and cross track errors ($\delta_1, \delta_2, \delta_3$) are shown in Figure 3. Note that although the perturbation solution produces only a small improvement in the radial error, this error is negligible in comparison to the down track error.

7 Conclusions

Our solution embodies the principles outlined in the introduction. The relative error of our solution is of order $(\theta - \theta_0)J^2$, which is a factor of J times the relative error of the two-body solution; our solution loses its validity after an angular change $(\theta - \theta_0)$ of order $1/J^2$, which is a factor of $\frac{1}{J}$ longer than the interval of validity of the two-body solution. Secondly, our solution is in terms of classical orbital elements; no transformation to an alternative non-physical set of elements is required. Finally, our solution is free of singularities for all values of the initial orbital parameters, including elliptic, parabolic, and hyperbolic orbits.

Our formulas should agree closely with satellite orbits whose dominant perturbation is the planet's oblateness. Of course, the effects of higher-order terms in these expansions, higher-order terms in the planet's potential, and of other perturbation forces may also be important. The formulas will have to be amended to include these additional effects.

APPENDIX I: Values of the Constants K_1 – K_6

$$K_1 = \frac{cse_0^2(-15s^2 + 14)}{24(5s^2 - 4)}$$

$$K_2 = -\frac{sc}{2} \cos 2\theta_0 - \frac{sce_0}{6} \cos(3\theta_0 - \omega_0) - \frac{sce_0}{2} \cos(\theta_0 + \omega_0) + \frac{cse_0^2(15s^2 - 14)}{24(5s^2 - 4)} \cos 2\omega_0$$

$$K_3 = \frac{e_0^2(-75s^6 + 260s^4 - 296s^2 + 112)}{48(5s^2 - 4)^2}$$

$$K_4 = e_0 \frac{[15(e_0^2 + 2)s^4 - 14(e_0^2 + 4)s^2 + 24]}{24(5s^2 - 4)}$$

$$\begin{aligned} K_5 = & \frac{e_0^2}{12}(-9s^2 + 8) \cos(2\theta_0 - 2\omega_0) + \frac{e_0^2}{24}(3s^2 - 2) \cos(4\theta_0 - 2\omega_0) \\ & -(e_0 s^2 + K_4) \cos(\theta_0 + \omega_0) + \frac{e_0}{8}(s^2 - 2) \cos(3\theta_0 - \omega_0) + \frac{e_0^2}{8}(-3s^2 + 2) \cos 2\omega_0 \\ & - \frac{1}{12}[5(2 - e_0^2)s^2 + 2e_0^2] \cos 2\theta_0 + \frac{1}{12}(15e_0^2 + 18)s^2 - (e_0^2 + 1) \end{aligned}$$

$$\begin{aligned} K_6 = & + \frac{e_0^2}{6}(6s^2 - 5) \sin(2\theta_0 - 2\omega_0) + \frac{e_0^2}{12}(-3s^2 + 1) \sin(4\theta_0 - 2\omega_0) \\ & + \frac{1}{2}[e_0(-s^2 + 1) + 2K_4] \sin(\theta_0 + \omega_0) + \frac{e_0}{2}(3s^2 - 2) \sin(\theta_0 - \omega_0) \\ & + \frac{e_0}{8}(-7s^2 + 2) \sin(3\theta_0 - \omega_0) + \frac{e_0^2}{4}(-s^2 + 1) \sin 2\omega_0 + \frac{1}{6}[-(5e_0^2 + 2)s^2 + 2e_0^2] \sin 2\theta_0 \end{aligned}$$

APPENDIX II: Rigorous Bounds on the Orbit

It follows from (10)–(12) that

$$T + V = \frac{1}{2} \left(\frac{dr}{dt} \right)^2 + \frac{r}{2} \frac{d^2 r}{dt^2} - \frac{GM}{2r} + \frac{GMJ_2 R^2}{4r^3} (1 - 3 \sin^2 \beta)$$

This can be rewritten in the form

$$\frac{d}{dr} \left[r^2 \left(\frac{dr}{dt} \right)^2 \right] = 4(T + V)r + 2GM + \frac{GMJ_2 R^2}{r^2} (3 \sin^2 \beta - 1)$$

from whence it follows that

$$\frac{d}{dr} \left[r^2 \left(\frac{dr}{dt} \right)^2 \right] \leq 4(T + V)r + 2GM + \frac{2GMJ_2 R^2}{r^2}$$

Integrating from $r(t_0)$ to $r(t)$ yields

$$r^2 \left(\frac{dr}{dt} \right)^2 \leq 2(T + V)r^2 + 2GMr - \frac{2GMJ_2 R^2}{r} - h_0^2 + \frac{3h_0^2 J_2 R^2}{p_0 r_0}$$

It follows that

$$0 < 2(T + V)r^2 + 2GMr - h_0^2 \left[1 - \frac{3J_2 R^2}{p_0 r_0} \right] \quad (44)$$

When $T + V < 0$, the quadratic polynomial on the right side of (44) has the roots (exact values can be found from the quadratic formula)

$$r_{\min} = \frac{p_0}{1 + e_0}[1 + O(J_2)], \quad r_{\max} = \frac{p_0}{1 - e_0}[1 + O(J_2)]$$

Hence a satellite having negative mechanical energy remains for all time within the spherical annulus $r_{\min} < r < r_{\max}$. Since the position vector is bounded, we can invoke the recurrence theorem; i.e., the satellite will come as close as desired to its initial position in a sufficiently long period of time (as shown by Poincaré). Furthermore, we are guaranteed of the validity of suppressing secular terms to describe the orbit via perturbation analysis.

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Figure 1: Orbital geometry.

Figure 2: Comparison of perturbation, two-body, and numerical orbits.

Figure 3: Comparison of perturbation, two-body, and measured orbits.